Parallel hp-FEM
hp-adaptive, hybrid-GMG, MatrixFree

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What is hp-adaptive FEM?

• Align mesh resolution with complexity of current solution
  - **h-adaptation:** dynamic cell sizes
    - good for irregular solutions
  - **p-adaptation:** dynamic function spaces
    - good for smooth solutions

• Combination of both possible

• *Algebraic* convergence with h-adaptation
• *Exponential* convergence possible with hp-adaptation
Timeline of hp-implementation in deal.II

- **2007/09/07**
  - deal.II 6.0: hp::DoFHandler, step-27

- **2009/04/27**
  - deal.II 6.2: FE Nothing

- **2010/06/25**
  - deal.II 6.3:

- **2011/10/09**
  - deal.II 7.1: step-46

- **2017/04/06**
  - deal.II 8.5: FESeries, step-27 rework

- **2018/05/11**
  - deal.II 9.0: hp::DoFHandler with shared::Triangulation

- **2019/05/21**
  - deal.II 9.1: hp::DoFHandler with distributed::Triangulation

- **2020/05/20**
  - deal.II 9.2: hp::Refinement, future FE indices

- **2021/06/01**
  - deal.II 9.3: MGTransferGlobalCoarsening, step-75, step-27 rework
How to use hp in deal.II?

- Enable hp-mode in DoFHandler with hp::FECollection
  - Note: hp::DoFHandler no longer required
- active_fe_index sets FE on each cell

- future_fe_index determines FE on each cell after refinement
- hp::Refinement namespace offers decision strategies
- Decision indicators via SmoothnessEstimator, predict_error

Serial example

step-27 demonstrates a basic serial hp-application.
Load balancing

- Partition domain into subdomains for parallelization
- Balance workload on all subdomains
- Workload on each cell proportional to number of DoFs
- Requires *weighted repartitioning* for load balancing
  - `parallel::CellWeights`
Efficient solvers

- System matrix with many nonzero entries for high-order FE
- Number of nonzero entries per row varies with hp-FEM
- Pure algebraic multigrid (AMG) methods struggle with irregular matrices
  - Number of solver iterations explode with increasing fragmentation

- Alternative: Geometric multigrid (GMG) methods
- Hybrid-GMG: Combination of p-multigrid, h-multigrid, and AMG
- Possible with new global coarsening algorithms
  - MGTransferGlobalCoarsening
Hybrid-GMG

$p$-restrict

$p$-prolong

$h$-restrict

$h$-prolong

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MatrixFree

- Memory bandwidth is bottleneck in HPC-applications
- Avoid storing system matrices
  - MatrixFree

- Hybrid-GMG allows us to use that feature

Parallel example

step-75 demonstrates load balancing, hybrid-GMG, and MatrixFree methods combined in the hp-context.
Example: Laplace problem, step-75

- Singularity at reentrant corners for elliptic problems
- L-shaped domain:
  \[
  \Omega = [-1, 1]^2 \setminus ([0, 1] \times [-1, 0])
  \]
- Manufactured Laplace problem
  \[
  -\nabla^2 u = 0 \quad \text{on} \quad \Omega \\
  u = \bar{u} \quad \text{on} \quad \partial \Omega \\
  \bar{u} = r^{2/3} \sin \left( \frac{2}{3} \varphi \right) \\
  \|
  \nabla \bar{u} \|
  = r^{-1/3}
  \]

Figure: L-shaped domain
Results: Cycle 0

Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 1

Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 2

**Figure: Polynomial degrees**

**Figure: Partitioning**
Results: Cycle 3

Figure: Polynomial degrees

Figure: Partitioning
Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 5

Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 6

Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 7

Figure: Polynomial degrees

Figure: Partitioning
Results: Cycle 7 (zoom x20)

Figure: Polynomial degrees

Figure: Partitioning
Results: Error convergence

Figure: Error performances of several adaptation strategies

1 M. Fehling, Algorithms for massively parallel generic hp-adaptive finite element methods

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Results: Strong scaling

Figure: Scaling on consecutively refined meshes for 768 MPI processes and a pure AMG solver\(^1\)

\(^1\)M. Fehling, Algorithms for massively parallel generic hp-adaptive finite element methods