

Theory: Linear elastic active muscle model

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An introduction to the theory applied to the linear elastic active muscle model of the biceps brachii.

1 Governing equations for quasi-static linear elasticity

The strong statement of the balance of linear momentum reads

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \text{on } \Omega \quad , \quad (1)$$

where $\nabla = \frac{\partial}{\partial x}$ is a differential operator, $\boldsymbol{\sigma}$ is the Cauchy stress tensor and $\mathbf{b} = \rho \mathbf{g}$ is the body force density vector. This is expressed in index notation as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0 \quad \text{on } \Omega \quad . \quad (2)$$

Pre-multiplying the above by test function $\delta \mathbf{v}$ and integrating over the domain Ω renders

$$- \int_{\Omega} \delta v_i \frac{\partial \sigma_{ij}}{\partial x_j} dv = \int_{\Omega} \delta v_i b_i dv \quad (3)$$

that, by using the product rule for derivatives (i.e. integration by parts), becomes

$$\int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \sigma_{ij} dv - \int_{\Omega} \frac{\partial}{\partial x_j} [\delta v_i \sigma_{ij}] dv = \int_{\Omega} \delta v_i b_i dv \quad . \quad (4)$$

Finally, by applying divergence theorem to the second term in the above, we attain the weak form of the balance of linear momentum

$$\int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \sigma_{ij} dv = \int_{\Omega} \delta v_i b_i dv + \int_{\partial \Omega} \delta v_i \underbrace{\sigma_{ij} n_j}_{\bar{\mathbf{t}}_i} da \quad , \quad (5)$$

wherein \mathbf{n} represents the outward facing normal on $\partial \Omega$, the boundary of the domain, and $\bar{\mathbf{t}}$ the prescribed traction on the Neumann boundary.

2 Constitutive law: A linearised Hill three-element active muscle model with surrounding matrix [1]

The linear constitutive law used to model active muscle tissue is derived by [1] from the nonlinear model developed by [2, 3]. In the representation given here, we deviate slightly from the notation given in [1] to facilitate its implementation.

Embedding of one-dimensional fibre model into three-dimensional space

We begin by defining the decomposition of the Cauchy stress tensor into a matrix and fibre contribution as

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_m + \boldsymbol{\sigma}_f \quad (6)$$

where m, f respectively denote contributions from the surrounding matrix and muscle fibres. The isotropic linear constitutive law for the matrix surrounding the muscle fibres is

$$\boldsymbol{\sigma}_m = \mathbb{C}_m : \boldsymbol{\varepsilon} \quad (7)$$

where \mathbb{C}_m is the stiffness tensor for the matrix, and the small strain tensor

$$\boldsymbol{\varepsilon} = \frac{1}{2} \left[\nabla \mathbf{u} + [\nabla \mathbf{u}]^T \right] \quad (8)$$

The fibre stress and strain are computed by

$$\boldsymbol{\sigma}_f = T_f \mathbf{m} \otimes \mathbf{m} \quad , \quad \boldsymbol{\varepsilon}_f = [\mathbf{m} \otimes \mathbf{m}] : \boldsymbol{\varepsilon} \quad (9)$$

Linearised version of Martin's one-dimensional muscle model

Figure 1 shows an analogue for the sarcomere, the smallest building-block of active muscle fibres. The distributions of strains and stresses within the various elements of the

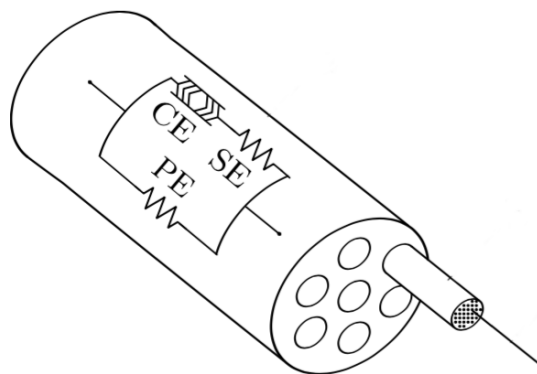


Figure 1: Schematic of the Hill-type muscle fibre [1].

representative model is determined by their arrangement with respect to one another. In the linearised (small-strain) version of the Hill three-element model, the decomposition of stress in the fibre as a whole and the one parallel branch are

$$T_f = T_p + T_s \quad \text{and} \quad T_c = T_s \quad (10)$$

where T is a measure of nominal stress, and the subscripts f, p, s, c respectively denote the fibre (as a whole), and the parallel, series and contractile element in the Hill model. Similarly, the decomposition of the (small) strains in the Hill model are

$$\varepsilon_f = \varepsilon_p \equiv \varepsilon_s + \varepsilon_c \quad . \quad (11)$$

The constitutive laws governing the response of each element are as follows:

$$T_p = T_0 f_p \quad , \quad T_s = T_0 f_s \quad \text{and} \quad T_c = f_c^l(\varepsilon_c) f_c^v(\dot{\varepsilon}_c) \alpha(u(t)) \quad . \quad (12)$$

where T_0 is the nominal stress, a physiological constant which defines to the maximum force of contraction under isometric conditions. Here the driver functions for the passive parallel and series elements are

$$f_p(\varepsilon_f) = \begin{cases} m_p \varepsilon_f & \text{if } \varepsilon_f > 0 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

$$f_s(\varepsilon_s) = \begin{cases} m_s \varepsilon_s \equiv m_s [\varepsilon_f - \varepsilon_c] & \text{if } \varepsilon_s \equiv \varepsilon_f - \varepsilon_c > 0 \\ 0 & \text{otherwise} \end{cases} \quad . \quad (14)$$

Here the strain relationship between the elements is used to remove ε_s as an unknown. For the active contractile element, the force-length and force-velocity relationships are approximated as

$$f_c^l(\varepsilon_c) = \begin{cases} 1 & \text{if } -0.5 \leq \varepsilon_c \leq 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

$$f_c^v(\dot{\varepsilon}_c) = \begin{cases} 0 & \text{if } \dot{\varepsilon}_c < -5 \\ \frac{1}{5}\dot{\varepsilon}_c + 1 & \text{if } -5 \leq \dot{\varepsilon}_c < 3 \\ 1.6 & \text{otherwise} \end{cases} \quad , \quad (16)$$

the latter of which we can write in general as

$$f_c^v(\dot{\varepsilon}_c) = m_c^v \dot{\varepsilon}_c + c_c^v \quad . \quad (17)$$

Note that alternative linearisations for these terms are possible, and that the rate-dependence of the contractile element makes this model “visco-elastic”. The differential equation that defines the muscle activation model [4] is expressed a function of the neural signal $u(t)$ by

$$\dot{\alpha}(u(t)) = \frac{1}{\tau_r} [1 - \alpha] u + \frac{1}{\tau_f} [\alpha_{\min} - \alpha] [1 - u] \quad . \quad (18)$$

The parameters τ_r and τ_f control the rise and fall of the activation function with respect to the history of the neural signal, and α_{\min} is the minimum activation level (real muscles are never completely inactive; they always retain some degree of tetanisation).

Time differentiation

For all time derivatives we employ a first-order backward Euler scheme. Therefore the contractile strain rate and rate of change of muscle activation at timestep n are approximated as

$$\dot{\varepsilon}_c \approx \frac{\varepsilon_c^n - \varepsilon_c^{n-1}}{\Delta t} \quad (19)$$

$$\dot{\alpha} \approx \frac{\alpha^n - \alpha^{n-1}}{\Delta t} . \quad (20)$$

Consequently the expression for the force-velocity relationship and activation level can be explicitly stated in terms of the history variables $\varepsilon_c^{n-1}, \alpha^{n-1}$ and the remaining unknowns $\varepsilon_c^n, \alpha^n$.

Substitution of fibre constitutive laws into one-dimensional stress relationship

From the equivalence of T_c and T_s , substituting in all of the salient previously derived expressions and considering $\alpha > 0$, we can extract the explicit expression for ε_c in terms of ε_f by the following steps:

$$\begin{aligned} f_c^l f_c^v \alpha &= f_s \\ \Rightarrow f_c^l \left[m_c^v \frac{\varepsilon_c - \varepsilon_c^{n-1}}{\Delta t} + c_c^v \right] \alpha &= m_s [\varepsilon_f - \varepsilon_c] \end{aligned}$$

that, with some further rearrangement, becomes

$$\begin{aligned} \varepsilon_c &= \underbrace{\left[f_c^l m_c^v \frac{1}{\Delta t} \alpha + m_s \right]}_{\beta}^{-1} \left[m_s \varepsilon_f + \underbrace{f_c^l \alpha \left[m_c^v \varepsilon_c^{n-1} \frac{1}{\Delta t} - c_c^v \right]}_{\gamma} \right] \\ &= \frac{m_s}{\beta} \varepsilon_f + \frac{\gamma}{\beta} \end{aligned} \quad (21)$$

Note that $\beta > 0$ under all conditions as $m_s > 0$ during contraction.

Substitution of constitutive laws into three-dimensional stress relationship

For the most general case, we can decompose the total Cauchy stress as

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbb{C}_m : \boldsymbol{\varepsilon} + \boldsymbol{\sigma}_f \\
&= \mathbb{C}_m : \boldsymbol{\varepsilon} + T_f \mathbf{m} \otimes \mathbf{m} \\
&= \mathbb{C}_m : \boldsymbol{\varepsilon} + T_0 [m_p \varepsilon_f + m_s [\varepsilon_f - \varepsilon_c]] \mathbf{m} \otimes \mathbf{m} \\
&= \mathbb{C}_m : \boldsymbol{\varepsilon} + T_0 \left[m_p \varepsilon_f + m_s \left[\varepsilon_f - \left[\frac{m_s}{\beta} \varepsilon_f + \frac{\gamma}{\beta} \right] \right] \right] \mathbf{m} \otimes \mathbf{m} \\
&= \mathbb{C}_m : \boldsymbol{\varepsilon} + T_0 \left[m_p + m_s - \frac{m_s^2}{\beta} \right] \varepsilon_f \mathbf{m} \otimes \mathbf{m} - \left[T_0 m_s \frac{\gamma}{\beta} \right] \mathbf{m} \otimes \mathbf{m} \\
&= \left[\underbrace{\mathbb{C}_m + T_0 \left[m_p + m_s - \frac{m_s^2}{\beta} \right] \mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m} \otimes \mathbf{m}}_{\mathbb{C}_f^*} : \boldsymbol{\varepsilon}(\mathbf{u}) - \underbrace{\left[T_0 m_s \frac{\gamma}{\beta} \right] \mathbf{m} \otimes \mathbf{m}}_{\boldsymbol{\sigma}_f^*} \right].
\end{aligned} \tag{22}$$

Note here that the first term on the right hand side ($[\mathbb{C}_m + \mathbb{C}_f^*] : \boldsymbol{\varepsilon}(\mathbf{u})$) is dependent on the solution, and the second term ($\boldsymbol{\sigma}_f^*$) depends only on local history variables.

3 Finite element discretisation

Combining eqs. (5) and (22) renders the complete expression of the balance of linear momentum, with accommodation of the muscle fibre model, namely

$$\int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} [\mathbb{C}_m + \mathbb{C}_f^*]_{ijkl} \varepsilon_{kl} dv = \int_{\Omega} \delta v_i b_i dv + \int_{\partial \Omega} \delta v_i \underbrace{\sigma_{ij} n_j}_{\bar{t}_i} da - \int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} [\boldsymbol{\sigma}_f^*]_{ij} dv, \tag{23}$$

We discretise the trial solution and test function using finite element shape functions (ansatz)

$$\mathbf{u}(\mathbf{x}) \approx \sum_I \boldsymbol{\Phi}^I(\mathbf{x}) u^I, \quad \mathbf{v}(\mathbf{x}) \approx \sum_I \boldsymbol{\Phi}^I(\mathbf{x}) v^I \tag{24}$$

where $\mathbf{N}^I(\mathbf{x})$ is the (position-dependent) vector-valued finite element shape function corresponding to the I^{th} degree-of-freedom, and u^I, v^I are coefficients of the solution and trial function. In `deal.II` nomenclature, the shape function is computed from a scalar base shape function and some expansion into higher-dimensional space by

$$\boldsymbol{\Phi}^I(\mathbf{x}) = N^I(\mathbf{x}) \mathbf{e}_{\text{comp}(I)} \tag{25}$$

where N^I is a scalar shape function and $\mathbf{e}_{\text{comp}(I)}$ is the basis direction associated with the I^{th} degree-of-freedom. Therefore, the j^{th} local component of shape function $\Phi^I(\mathbf{x})$ is given by

$$[\Phi^I(\mathbf{x})]_j = N^I(\mathbf{x}) [\mathbf{e}_{\text{comp}(I)}]_j = N^I(\mathbf{x}) \delta_{\text{comp}(I)j} \quad . \quad (26)$$

where δ_{ij} is the Kronecker delta. Note that in this instance we use the same ansatz for the test and trial spaces, and the $0 \leq \text{comp}(I), j < \text{spacedim}$.

We now use these shape functions to discretise the weak expression for the balance of linear momentum. Starting on the right-hand side of eq. (23), the body force and traction contributions are computed by

$$\begin{aligned} \int_{\Omega} \delta v_i b_i dv &= \int_{\Omega} \left[\sum_I \Phi^I(\mathbf{x}) \delta v^I \right]_i b_i dv = \sum_I \delta v^I \int_{\Omega} [\Phi^I(\mathbf{x})]_i b_i dv \\ &= \sum_I \delta v^I \int_{\Omega} N^I(\mathbf{x}) \delta_{\text{comp}(I)i} b_i dv = \sum_I \delta v^I \int_{\Omega} N^I b_{\text{comp}(I)} dv \end{aligned} \quad (27)$$

$$\int_{\Omega} \delta v_i t_i dv = \sum_I \delta v^I \int_{\Omega} N^I t_{\text{comp}(I)} dv \quad . \quad (28)$$

while the contribution to the right-hand side that arise from the history variables is

$$\begin{aligned} - \int_{\Omega} \frac{\partial}{\partial x_j} [\delta v_i] [\sigma_f^*]_{ij} dv &= - \int_{\Omega} \frac{\partial}{\partial x_j} \left[\sum_I \Phi^I(\mathbf{x}) \delta v^I \right]_i [\sigma_f^*]_{ij} dv \\ &= - \sum_I \delta v^I \int_{\Omega} \frac{\partial}{\partial x_j} [\Phi^I(\mathbf{x})]_i [\sigma_f^*]_{ij} dv \\ &= - \sum_I \delta v^I \int_{\Omega} \frac{\partial}{\partial x_j} [N^I(\mathbf{x}) \delta_{\text{comp}(I)i}] [\sigma_f^*]_{ij} dv \\ &= - \sum_I \delta v^I \int_{\Omega} \frac{\partial N^I(\mathbf{x})}{\partial x_j} [\sigma_f^*]_{\text{comp}(I)j} dv \quad . \end{aligned} \quad (29)$$

The last component of eq. (23) that we wish to express in discrete form is the left-hand side of the equation. Before we do, we observe that using the minor symmetry of the material stiffness tensor we can re-express the contraction of it and the small strain tensor as

$$\mathbb{C} : \boldsymbol{\varepsilon} = \mathbb{C} : \frac{1}{2} \left[\nabla \mathbf{u} + [\nabla \mathbf{u}]^T \right] \equiv \mathbb{C} : \nabla \mathbf{u} \quad (30)$$

Therefore, this contribution written in discrete form is

$$\begin{aligned}
& \int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} [\mathbb{C}_m + \mathbb{C}_f^*]_{ijkl} \varepsilon_{kl} dv \equiv \int_{\Omega} \frac{\partial \delta v_i}{\partial x_j} \mathbb{C}_{ijkl} \frac{\partial \delta u_k}{\partial x_l} dv \\
& \equiv \int_{\Omega} \frac{\partial}{\partial x_j} \left[\sum_I \boldsymbol{\Phi}^I(\mathbf{x}) \delta v^I \right]_i \mathbb{C}_{ijkl} \frac{\partial}{\partial x_l} \left[\sum_J \boldsymbol{\Phi}^J(\mathbf{x}) \delta u^J \right]_k dv \\
& \equiv \sum_{I,J} \delta v^I \left[\int_{\Omega} \frac{\partial}{\partial x_j} [\boldsymbol{\Phi}^I(\mathbf{x})]_i \mathbb{C}_{ijkl} \frac{\partial}{\partial x_l} [\boldsymbol{\Phi}^J(\mathbf{x})]_k dv \right] \delta u^J \\
& \equiv \sum_{I,J} \delta v^I \left[\int_{\Omega} \frac{\partial N^I(\mathbf{x})}{\partial x_j} \delta_{\text{comp}(I)i} \mathbb{C}_{ijkl} \frac{\partial N^J(\mathbf{x})}{\partial x_l} \delta_{\text{comp}(J)k} dv \right] \delta u^J \\
& \equiv \sum_{I,J} \delta v^I \left[\int_{\Omega} \frac{\partial N^I(\mathbf{x})}{\partial x_j} \mathbb{C}_{\text{comp}(I)j \text{comp}(J)l} \frac{\partial N^J(\mathbf{x})}{\partial x_l} dv \right] \delta u^J . \quad (31)
\end{aligned}$$

Equations (27) to (29) and (31) are collectively used to develop the system of linear equations that are solved at each time step.

References

- [1] Y. Kajee, J-P. V. Pelteret, and B. D. Reddy. The biomechanics of the human tongue. *International Journal for Numerical Methods in Biomedical Engineering*, 29(4):492–514, April 2013.
- [2] J. A. C. Martins, M. P. M. Pato, and E. B. Pires. A finite element model of skeletal muscles. *Virtual and Physical Prototyping*, 1:159–170, 2006.
- [3] J. A. C. Martins, E. B. Pires, R. Salvado, and P. B. Dinis. A numerical model of passive and active behaviour of skeletal muscles. *Computer Methods in Applied Mechanics and Engineering*, 151:419–433, 1998.
- [4] M. G. Pandy, F. E. Zajac, E. Sim, and W. S. Levine. An optimal control model for maximum-height human jumping. *Journal of Biomechanics*, 23:1185–1198, 1990.