Unfitted Bulk Finite Element Method for Solving Surface Partial Differential Equations

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White Blood Cell Motion

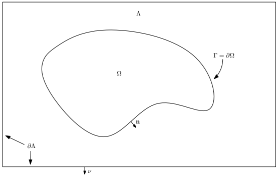
We model the motion of a free white blood cell in a liquid environment. The cell is represented implicitly as $\Omega = \{\mathbf{x} \in \Lambda \mid \varphi(\mathbf{x}, t) \geq 0\}$.

The Level Set Method + Incompressible Navier-Stokes

$$
\begin{cases}
\frac{\partial \varphi}{\partial t} + \mathbf{u} \cdot \nabla \varphi = 0 & \text{in } [0,T] \times \Lambda \\
\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \nabla \cdot (2\mu \nabla^s \mathbf{u}) + \nabla p = \mathbf{f} & \text{in } [0,T] \times \Lambda \\
\nabla \cdot \mathbf{u} = 0 & \text{in } [0,T] \times \Lambda \\
[2\mu \nabla^s \mathbf{u} - pI] \cdot \mathbf{n} = \mathbf{f}_\Gamma & \text{on } [0,T] \times \Gamma
\end{cases} \tag{1}
$$

with stabilization terms.

Here $\mathbf{f}_\Gamma$ represents the various physics that take place on the boundary of cell, for instance motion to minimize surface tension or minimize bending energy of membrane. In the more complicated cases $\mathbf{f}_\Gamma$ must be calculated as a solution to a geometric pde that lives on the manifold Γ.



Partial Differential Equation on Surface, Γ

Solve pde on bulk finite element with the mesh unfitted to the surface which is defined only implicitly such as by a level set method. In the case of Canham-Helfrich energy minimization in 2D (a simplified case), we need to be able to solve for

$$\mathbf{f}_\Gamma = k\left(\Delta_\Gamma H + \frac{1}{3}H^3\right), \qquad \mathbf{x} \in \Gamma \tag{2}$$

where H is the total curvature of the surface. Now, the vector curvature $H\mathbf{n}$ can be written as a scale multiple of $\Delta_\Gamma X$ where X is the identity operator on the surface, thus we study the surface Laplacian or Laplace-Beltrami equation.

Classical Laplace-Beltrami Equation

Find $u \in C^2(\Gamma)$ such that

$$-\Delta_\Gamma u + cu = f, \qquad \mathbf{x} \in \Gamma$$

Unfitted Bulk Finite Element Method

One direction of
research is in using a smeared Dirac delta function

$$\delta_\varepsilon(\mathbf{x}) = \begin{cases} \frac{1}{\varepsilon}\psi\left(\frac{\phi(\mathbf{x})}{\varepsilon}\right), & |\varphi(\mathbf{x})| < \varepsilon \\ 0, & \text{else} \end{cases} \tag{3}$$

with half-width
$\varepsilon = ch^\beta$ and kernel ψ. Sadly when $\beta = 1$, there are $\mathcal{O}(1)$ errors using δ_ε. But for example, with $\beta = 3/4$, we get $\mathcal{O}(h^{3/2})$ convergence using

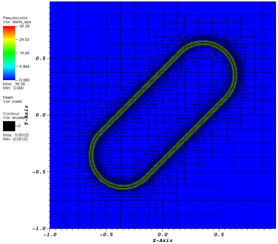


Figure : Example $\delta_\epsilon(\mathbf{x})$ for Γ = rotated capsule.

Unfitted Bulk Finite Element Method with Smeared Dirac Function

Find $u_h \in V_h$ such that for all $v_h \in V_h = \mathbb{P}^1(\mathcal{T}_h)$,

$$\int_\Omega \delta_\varepsilon(\mathbf{x})\left(\nabla u_h \cdot \nabla v_h + c u_h v_h\right)|\nabla\varphi_h| d\mathbf{x} = \int_\Omega \delta_\varepsilon(\mathbf{x}) f^e v_h |\nabla\varphi_h| d\mathbf{x}$$