

Numerical approximation of Riesz Fractional Laplacian equation

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Problem

Let Ω be the unit ball $B_1(0)$ and recall we are trying to use finite element method to solve the Riesz fractional Laplace equation

$$(-\Delta)^\beta u = f \text{ for } x \in \Omega$$

$$u = 0 \text{ for } x \in \Omega^c$$

where the Riesz fractional Laplace is defined by the Fourier transform

$$(-\hat{\Delta})^\beta f(\xi) = |\xi|^{2\beta} \hat{f}(\xi).$$

The basic idea to solve this equation in the weak form is to use the following Dunford-Taylor integral

$$((-\Delta)^\beta u, v) = \frac{2 \sin(\pi\beta)}{\pi} \int_0^\infty t^{-2\beta} (T_t u, v) \frac{dt}{t}.$$

Here u, v are in a certain space and $w = T_t u = -(I - t^2 \Delta)^{-1} t^2 \Delta u$ or w solves

$$(w, \phi) + t^2 (\nabla w, \nabla \phi) = t^2 (\nabla u, \nabla \phi) \text{ for all } \phi \in H^1(\mathbb{R}^d)$$

Results

We solve the equation

$$(-\Delta)^\beta u = 1 \text{ for } x \in \Omega$$

$$u = 0 \text{ for } x \in \Omega^c$$

and the analytic solution is

$$u(x) = 2^{-2\beta} \frac{\Gamma(d/2)}{\Gamma((d+2\beta)/2)\Gamma(1+\beta)} (1 - |x|^2)^\beta.$$

We test different β and the L^2 error is listed as follows

n	$\beta = 0.3$	order	$\beta = 0.8$	order
2	2.31×10^{-1}	-	7.18×10^{-2}	-
3	1.36×10^{-1}	0.76	3.60×10^{-2}	1.00
4	8.07×10^{-2}	0.75	1.81×10^{-2}	0.99
5	4.76×10^{-2}	0.76	9.04×10^{-3}	1.00

Future work

- ① Multilevel method need to be applied for inner and outer solver.
- ② Parallel computing